# Appendix: Orthogonal Curvilinear Coordinates

## Notes:

*Most of the material presented in this chapter is taken from Anupam, G. (Classical Electromagnetism in a Nutshell 2012, (Princeton: New Jersey)), Chap. 2, and Weinberg, S. (Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity 1972, (Wiley: New York), Chap. 8.*

We define the infinitesimal spatial displacement vector $d\mathbf{x}$ in a given orthogonal coordinate system with

$$d\mathbf{x} = dx^i \mathbf{e}_i, \tag{II.1}$$

where the Einstein summation convention was used, $dx^i$ is a contravariant component and $\mathbf{e}_i$ is a basis vector ($i = 1,2,3$). The length interval $ds$ is thus given by

$$\begin{aligned} ds^2 &= d\mathbf{x} \cdot d\mathbf{x} \\ &= \left(dx^i \mathbf{e}_i\right) \cdot \left(dx^j \mathbf{e}_j\right) \\ &= \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) dx^i dx^j \\ &= g_{ij} dx^i dx^j \\ &= dx^i dx_i, \end{aligned} \tag{II.2}$$

where the orthogonality of the coordinate system is specified by $\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij}$ with the metric tensor $g_{ij} = 0$ when $i \neq j$, and $dx_i$ is the covariant component. Please note that basis vectors are not unit vectors, i.e., $\mathbf{e}_i \cdot \mathbf{e}_i \neq 1$ in general. Equation (II.2) can be used to similarly define the inner product between any two vectors with

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= g_{ij} A^i B^j \\ &= A^i B_i. \end{aligned} \tag{II.3}$$

Since the covariant and contravariant components are generally different from one another in non-Cartesian coordinate systems, it is often more desirable to introduce a new set of so-called *ordinary* or *physical components* that preserve the inner product without explicitly bringing in both types of components or the metric tensor.

We start by rewriting equation (II.2) as

$$ds^2 = \left(h_1 du^1\right)^2 + \left(h_2 du^2\right)^2 + \left(h_3 du^3\right)^2 \tag{II.4}$$

for the orthogonal coordinate system $\left(u^1, u^2, u^3\right)$. A comparison with equation (II.2) reveals that $h_i^2 = g_{ii}$. For example, Cartesian coordinates have $h_1 = h_2 = h_3 = 1$, cylindrical coordinates $(\rho, \theta, z)$ have $h_1 = h_3 = 1$, $h_2 = \rho$, and spherical coordinates $(r, \theta, \phi)$ have $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin(\theta)$. Going back to equation (II.3) for the inner product, we now define the physical coordinates $\overline{A}_i$ of a vector $\mathbf{A}$ such that

$$\mathbf{A} \cdot \mathbf{B} \equiv \overline{A}_i \overline{B}_i, \tag{II.5}$$

where the use of subscripts has no particular meaning (i.e., a subscript does not imply a covariant component). A comparison with equation (II.3) implies that the physical components are related to the covariant and contravariant components through

$$\overline{A}_i = h_i A^i = h_i^{-1} A_i. \tag{II.6}$$

The first thing we should notice is that the physical components allow the use of a unit basis $\hat{\mathbf{e}}_i$ since

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}_j &= g_{ij} \\ &= h_i h_j \delta_{ij} \\ &= h_i h_j \left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j\right) \\ &= \left(h_i \hat{\mathbf{e}}_i\right) \cdot \left(h_j \hat{\mathbf{e}}_j\right). \end{aligned} \tag{II.7}$$

In fact, we could have alternatively justified the introduction of the physical components by the desire to use a unit basis with

$$\begin{aligned} d\mathbf{x} &= h_1 du^1 \hat{\mathbf{e}}_1 + h_2 du^2 \hat{\mathbf{e}}_2 + h_3 du^3 \hat{\mathbf{e}}_3 \\ &= d\overline{x}_1 \hat{\mathbf{e}}_1 + d\overline{x}_2 \hat{\mathbf{e}}_2 + d\overline{x}_3 \hat{\mathbf{e}}_3 \end{aligned} \tag{II.8}$$

or in general

$$\mathbf{A} = \overline{A}_1 \hat{\mathbf{e}}_1 + \overline{A}_2 \hat{\mathbf{e}}_2 + \overline{A}_3 \hat{\mathbf{e}}_3. \tag{II.9}$$

It should now be clear that what we usually specify as *coordinates* (e.g., $(\rho, \theta, z)$ and $(r, \theta, \phi)$) correspond to the contravariant components of $d\mathbf{x}$, while the *physical coordinates* are those for which the components of $d\mathbf{x}$ have units of length (e.g., $(d\rho, \rho d\theta, dz)$ and $(dr, r d\theta, r\sin(\theta) d\phi)$).

We now define the different differential operators using the physical coordinates, starting with the gradient. To do so, we first consider the differential of a scalar function $f$

$$\begin{aligned} df &= \frac{\partial f}{\partial u^i} du^i \\ &\equiv \nabla f \cdot d\mathbf{x} \\ &= \nabla f \cdot \left( \sum_i h_i du^i \hat{\mathbf{e}}_i \right) \\ &= \sum_i h_i du^i \nabla f \cdot \hat{\mathbf{e}}_i , \end{aligned} \tag{II.10}$$

which from the first and last equations implies that

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u^1} \hat{\mathbf{e}}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u^2} \hat{\mathbf{e}}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u^3} \hat{\mathbf{e}}_3 . \tag{II.11}$$

This leads to the following relations for the cylindrical and spherical coordinate systems

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial \rho} \hat{\mathbf{e}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z} \hat{\mathbf{e}}_z \\ \nabla f &= \frac{\partial f}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \phi} \hat{\mathbf{e}}_\phi , \end{aligned} \tag{II.12}$$

respectively. For the divergence of a vector we consider an infinitesimal cube, as shown in Figure 1, and use the divergence theorem

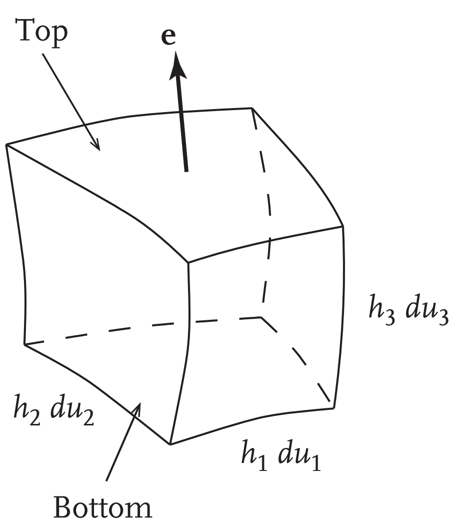


**Figure 1 -** Infinitesimal volume of integration, where we do not differentiate between $du^i$ and $du_i$.

$$
\begin{aligned}
\int_V \nabla\cdot\mathbf{A}d^3x &= \nabla\cdot\mathbf{A}h_1h_2h_3du^1du^2du^3 \\
&= \int_S \mathbf{A}\cdot\mathbf{n}\,da,
\end{aligned}
\tag{II.13}
$$

which for a small enough cube we can write as

$$
\begin{aligned}
\int_S \mathbf{A}\cdot\mathbf{n}\,da &= \left[\bar{A}_1h_2h_3\right]_{\text{right}}^{\text{left}} du^2du^3 + \left[\bar{A}_2h_1h_3\right]_{\text{back}}^{\text{front}} du^1du^3 \\
&\quad + \left[\bar{A}_3h_1h_2\right]_{\text{bottom}}^{\text{top}} du^1du^2 \\
&= \left[\frac{\partial}{\partial u^1}\left(\bar{A}_1h_2h_3\right) + \frac{\partial}{\partial u^2}\left(\bar{A}_2h_1h_3\right) + \frac{\partial}{\partial u^3}\left(\bar{A}_3h_1h_2\right)\right]du^1du^2du^3,
\end{aligned}
\tag{II.14}
$$

since in general $h_i$ can vary across the dimensions of the cube. A comparison with equation (II.13) reveals that

$$
\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}\left(\bar{A}_1h_2h_3\right) + \frac{\partial}{\partial u^2}\left(\bar{A}_2h_1h_3\right) + \frac{\partial}{\partial u^3}\left(\bar{A}_3h_1h_2\right)\right]. \tag{II.15}
$$

We then respectively have for cylindrical and spherical coordinates

$$
\begin{aligned}
\nabla\cdot\mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\bar{A}_r\right) + \frac{1}{\rho}\frac{\partial\bar{A}_\theta}{\partial\theta} + \frac{\partial\bar{A}_z}{\partial z} \\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\bar{A}_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\bar{A}_\theta\right) + \frac{1}{r\sin(\theta)}\frac{\partial\bar{A}_\phi}{\partial\phi}.
\end{aligned}
\tag{II.16}
$$

The Laplacian is readily evaluated by setting $\mathbf{A} = \nabla f$ and inserting equations (II.12) in equations (II.16). We then have the corresponding relations

$$
\begin{aligned}
\nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2} \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}
\end{aligned}
\tag{II.17}
$$

for cylindrical and spherical coordinates, respectively.

Finally, for the curl we use Stokes' Theorem using an infinitesimal surface as shown in Figure 2

$$
\begin{aligned}
\int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,da &= (\nabla\times\mathbf{A})\cdot\left(\alpha h_2h_3du^2du^3\hat{\mathbf{e}}_1 + \beta h_1h_3du^1du^3\hat{\mathbf{e}}_2 + \gamma h_1h_2du^1du^2\hat{\mathbf{e}}_3\right) \\
&= \oint_C \mathbf{A}\cdot d\mathbf{l},
\end{aligned}
\tag{II.18}
$$

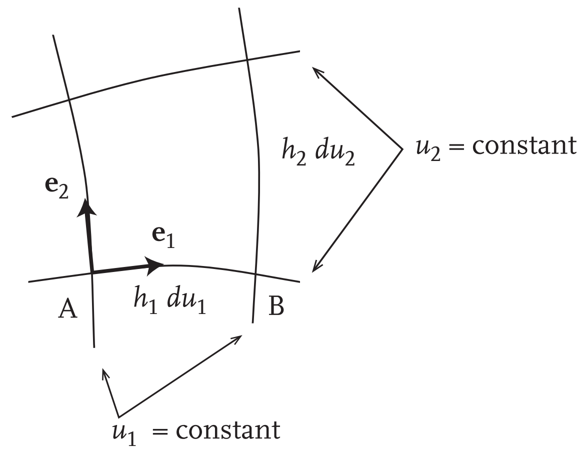


**Figure 2** – Infinitesimal loop of integration for the derivation of the curl, projection in the $(u^1, u^2)$-plane.

where $\mathbf{n} = \alpha\hat{\mathbf{e}}_1 + \beta\hat{\mathbf{e}}_2 + \gamma\hat{\mathbf{e}}_3$. For this infinitesimal loop we can consider the different projections on the three $(u^i, u^j)$-planes and write (using the first two terms of the corresponding Taylor expansions)

$$\begin{aligned}
\oint_C \mathbf{A}\cdot d\mathbf{l} &= \alpha\left\{\left[\bar{A}_2 h_2\right]_{\text{top}}^{\text{bottom}} du^2 + \left[\bar{A}_3 h_3\right]_{\text{back}}^{\text{front}} du^3\right\} \\
&\quad + \beta\left\{\left[\bar{A}_1 h_1\right]_{\text{bottom}}^{\text{top}} du^1 + \left[\bar{A}_3 h_3\right]_{\text{left}}^{\text{right}} du^3\right\} \\
&\quad + \gamma\left\{\left[\bar{A}_1 h_1\right]_{\text{front}}^{\text{back}} du^1 + \left[\bar{A}_2 h_2\right]_{\text{right}}^{\text{left}} du^2\right\} \\
&= \alpha\left[-\frac{\partial}{\partial u^3}\left(\bar{A}_2 h_2\right) + \frac{\partial}{\partial u^2}\left(\bar{A}_3 h_3\right)\right] \\
&\quad + \beta\left[\frac{\partial}{\partial u^3}\left(\bar{A}_1 h_1\right) - \frac{\partial}{\partial u^2}\left(\bar{A}_3 h_3\right)\right] \\
&\quad + \gamma\left[-\frac{\partial}{\partial u^2}\left(\bar{A}_1 h_1\right) + \frac{\partial}{\partial u^1}\left(\bar{A}_2 h_2\right)\right].
\end{aligned} \tag{II.19}$$

Equating equations (II.18) and (II.19) we must have

$$\begin{aligned}
\nabla\times\mathbf{A} &= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u^2}\left(\bar{A}_3 h_3\right) - \frac{\partial}{\partial u^3}\left(\bar{A}_2 h_2\right)\right]\hat{\mathbf{e}}_1 + \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial u^3}\left(\bar{A}_1 h_1\right) - \frac{\partial}{\partial u^1}\left(\bar{A}_3 h_3\right)\right]\hat{\mathbf{e}}_2 \\
&\quad + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u^1}\left(\bar{A}_2 h_2\right) - \frac{\partial}{\partial u^2}\left(\bar{A}_1 h_1\right)\right]\hat{\mathbf{e}}_3.
\end{aligned} \tag{II.20}$$

We then respectively write for the cylindrical and spherical coordinate systems

$$\nabla \times \mathbf{A} = \left[ \frac{1}{\rho} \frac{\partial \bar{A}_z}{\partial \theta} - \frac{\partial \bar{A}_\theta}{\partial z} \right] \hat{\mathbf{e}}_\rho + \left[ \frac{\partial \bar{A}_\rho}{\partial z} - \frac{\partial \bar{A}_z}{\partial \rho} \right] \hat{\mathbf{e}}_\theta + \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} \left( \rho \bar{A}_\theta \right) - \frac{\partial \bar{A}_\rho}{\partial \theta} \right] \hat{\mathbf{e}}_z$$

$$\nabla \times \mathbf{A} = \frac{1}{r \sin(\theta)} \left[ \frac{\partial}{\partial \theta} \left( \sin(\theta) \bar{A}_\phi \right) - \frac{\partial \bar{A}_\theta}{\partial \phi} \right] \hat{\mathbf{e}}_r + \left[ \frac{1}{r \sin(\theta)} \frac{\partial \bar{A}_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{A}_\phi \right) \right] \hat{\mathbf{e}}_\theta \quad \text{(II.21)}$$

$$+ \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \bar{A}_\theta \right) - \frac{\partial \bar{A}_r}{\partial \theta} \right] \hat{\mathbf{e}}_\phi .$$